Singleton Congestion Games

- Singleton Congestion game is a fundamental class of games with negative externality.

Agents (drivers) $[N] = \{1, \ldots, N\}$
Resource (locations with orders) $[R] = \{1, \ldots, R\}$
Non-decreasing congestion functions (orders each agent receives)
- $c_{\text{theater}}(n) = 2$
- $c_{\text{stadium}}(n) = \begin{cases} 0, & \text{if } n < N \\ 1, & \text{if } n = N \end{cases}$
- $c_{\text{grocery}}(n) = 1 + \epsilon$

Each agent selects a single resource $a_i^*$ that minimizes congestion from his available actions $A_i \subseteq [R]$.

Singleton Congestion Games w. Uncertainty

The congestion functions $c_i$ depend on a common random state of nature $\theta$ drawn from discrete support $\Theta$ with prior distribution $\mu \in \Delta_\Theta = \{ p \in \mathbb{R}_+^\Theta, \sum_{\Theta} p_\theta = 1 \}$

Game day $\theta_1$:
- $\mu_{\theta_1} = 0.6$
- $c_{\text{theater}}(n) = \begin{cases} 1.2, & \text{if } n < N \\ 1.6, & \text{if } n = N \end{cases}$
- $c_{\text{stadium}}(n) = \begin{cases} 0.8, & \text{if } n < N \\ 1.4, & \text{if } n = N \end{cases}$
- $c_{\text{grocery}}(n) = 1 + \epsilon$

Theorom 1. The social-cost minimizing public signaling scheme for SCGs, under optimistic equilibrium selection, can be computed in $\text{poly}(N, 2^{|R|-1})$ time.

The Blessing of Constant Resources

Optimal Public Signaling

Theorem 1. The social-cost minimizing public signaling scheme for SCGs, under optimistic equilibrium selection, can be computed in $\text{poly}(N, 2^{|R|-1})$ time.

Reduce to polynomial space

$a \in [R]^N$: the profile of numbers of agents choosing each resource
$\Lambda \in \{0, 1\}^{R \times (\text{poly}(R))}$: agents' preference over resources

Convert back with maximum many-to-one bipartite matching

Optimal Private Signaling

Theorem 2. The social-cost minimizing private signaling scheme can be computed in $\text{poly}(N^R)$ time.

Reduce to polynomial space

$\{x_{\text{init}}\}$: the marginal probability that conditioned on the state of nature is $\theta$, agent $i$ is assigned to resource $r$ and the configuration of all resources is $n$.

Convert to signaling scheme $\pi$ with max flow and flow decomposition

Hardness of SCGs with Many Resources

Equilibrium-Oblivious Intractability of Public Signaling

Theorem 3. It is equilibrium-obliviously NP-hard to obtain a $(1 + \frac{1}{\text{poly}(N)})$-approximation algorithm for the social-cost minimizing optimal public signaling in SCGs, even when agents have symmetric action sets.

Evidence of Intractability for Optimal Private Signaling

Proposition 1. It is NP-hard to solve the following Optimization Problem (separation oracle of the dual Linear Program), even when all resources have the same linearly increasing congestion functions $c_r(n) = n$ and the game is symmetric.

$$\min_{n \in \text{PP}(A)} \sum_{r \in [N]} n_r c_r(n_r) - \sum_{r \in [N]} \sum_{i \in A_r} \sum_{r' \in [N]} \left[ c_{r'}(n_{r'} + 1) - c_r(n_r) \right] \cdot z_{r,r'}$$