On the Robustness of Second-Price Auctions in Prior-Independent Mechanism Design

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Motivation

- mech design: how to optimally sell things
- classical theory too detail-dependent, so we **relax the common prior assumption** (Wilson doctrine)

Problem Formulation

- Optimize over direct mechanisms \((x, p)\) selling one indivisible item to \(n\) buyers.
- mechanism is **prior-independent**
  - no need to know \(F\) (“detail-free” or “robust”)
  - performance guarantee over all \(F \in \mathcal{F}\)
- We consider many dist classes on \([0,1]^n\).
- **dominant strategy** IC+IR
  - each buyer need not know other buyers’ dists
  - Objective = “regret” on revenue
  - Benchmark = maximum possible revenue when valuation is known = \(\max(v)\).

Research question:
What is an optimal detail-free mechanism and how well can we perform?

Challenges

- The space of all mechanisms is large.
- The space of all bounded dists is large.
- The problem is **nonconvex** due to class restriction in \(\mathcal{F}\) e.g. i.i.d.

Minimax Problem For Each Distribution Class \(\mathcal{F}\)

\[
\min_{\text{mech } (x,p)} \max_{F \in \mathcal{F}} \mathbb{E}_{v \sim F} \left[ \max(v) - \sum_{i=1}^{n} p_i(v) \right] 
\]

Main Result

- exchangeable and affiliated distributions \(\subset \) i.i.d. distributions
- mixture of i.i.d. distributions \(\subset \) all distributions

**Second Price Auction with Random Reserve** is minimax optimal across many distribution classes

Theorem

Under the distribution class of \(\{\text{i.i.d., mixture of i.i.d., exchangeable and affiliated}\}\), the minimax regret admits as an optimal mechanism a second-price auction with random reserve price with cumulative distribution \(\Phi_n^*\) on \([r_n, 1]\) given by

\[
\Phi_n^*(v) = \left( \frac{v}{v - r_n} \right)^{n-1} \log \left( \frac{v}{r_n} \right) - \sum_{k=1}^{n-1} \left( \frac{v}{v - r_n} \right)^{n-1-k} 
\]

where \(r_n \in (0, 1/n)\) is the unique solution to

\[
(1 - r^*)^{n-1} + \log(r^*) + \sum_{k=1}^{n-1} \frac{(1 - r^*)^k}{k} = 0.
\]

Our Approach

- **saddle point approach**: find \(m^*, F^*\)
  \(R(m^*, F^*) \leq R(m^*, F^*) \leq R(m, F^*) \quad \forall m, F\)
- Conjecture that \(m^* = \text{SPA}(\Phi^*)\).
- \(\Phi^*\) minimizes \(R(\Phi, F^*)\), which is linear in \(\Phi \Rightarrow \text{pins down } F^*\)
- \(F^*\) maximizes \(R(\Phi^*, F)\) which is a function of \(F(\cdot) \Rightarrow \text{pins down } \Phi^*\)

Insights

- **reserve CDF as a function of \(n\)**

<table>
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<th>(n)</th>
<th>OPT</th>
<th>SPA(0)</th>
<th>SPA((r^*))</th>
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</table>

- value of competition positive as \(n \to \infty\)
- significantly outperforms benchmarks (no & optimal deterministic reserves)