Approximately Strategyproof Tournament Rules with Multiple Prizes

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**Setup**

- A tournament $T$ consists of a set of $n$ teams as well as the results of all $\binom{n}{2}$ matches among all pairs of teams.
- A tournament ranking rule $r$ is a function that maps tournaments $T$ to a distribution over rankings $\sigma$ (where $\sigma(v)$ represents the ranking of team $v$).
- A prize vector is a non-increasing vector $\vec{p} \in \mathbb{R}^n$ such that the team ranked $j^\text{th}$ receives $p_j$ in prize money. In particular, the vector $\vec{p}$ with $p_j = \frac{1}{n-1}$ is called the Borda prize vector.

**Measures of Fairness**

- A team is a Condorcet winner of a tournament if it beats every other team. A tournament ranking rule is Condorcet-Consistent if it outputs a ranking where a Condorcet winner, if one exists, is always ranked first with probability 1.
- Team $i$ covers team $j$ if $i$ beats $j$, and $i$ beats every team that $j$ beats. A tournament ranking rule $r$ is Cover-Consistent if whenever $i$ covers $j$, $r$ outputs a ranking where $i$ is ahead of $j$ with probability 1.

**Manipulability**

- Let $S$ be a set of teams. Two tournaments $T, T'$ are $S$-adjacent if they are identical except for matches between two teams in $S$.
- We define $\alpha^k_r(T)$ to be the maximum prize money under $\vec{p}$ that any set of $k$ teams can manipulate in $r$ by manipulating the underlying tournament $T$ to an $S$-adjacent $T'$. For a class of prize vectors $P$, we define $\alpha^k_r(P)$ to be the maximum prize money that any set of $k$ teams can manipulate in $r$ under any $\vec{p} \in P$.
- We define $\alpha^k_r(P)$ to be the best bound on manipulability achievable by a Condorcet-Consistent tournament ranking rule against collisions of $k$ teams that holds for all prize vectors in $P$.

**Related Work / Motivation**

Previous authors ([1], [2]) have designed tournament rules with $\alpha^k_2 = 1/3$ for $\vec{p} = (1, 0, \ldots, 0)$. This has been shown to be the best possible result among all Condorcet-Consistent tournament rules. Some manipulability bounds also exist for $k > 2$. As illustrated above, all prize rules only consider the case where $\vec{p} = (1, 0, \ldots, 0)$. However, several modern tournaments offer rewards for teams beyond the winner. For example, the League of Legends Championship Series directly awards a monetary prize to teams based on their final ranking. Our work extends previous works to consider other different prize vectors and establish non-manipulability bounds.

**Nested Randomized King of the Hill (NRKotH)**

Consider a tournament $T$ on a set $S$ of teams. Let $\sigma(a)$ represent the rank of team $a$. We define the tournament rule NRKotH on this tournament $T$ on $n = |S|$ teams as follows:

1. If $n = 0$, return an empty ordering. Else, continue.
2. Pick a team, $u$, uniformly at random. Call $u$ the pivot.
3. Let $B$ denote the teams that beat $u$, and $L$ denote the teams that lose to $u$.
4. Run NRKotH on $B$ and $L$, and call the outputs $\sigma_B$ and $\sigma_L$, respectively.
5. For all teams $b \in B$, set $\sigma(b) = \sigma_B(b)$.
6. Set $\sigma(u) = |B| + 1$.
7. For all teams $l \in L$, set $\sigma(l) = \sigma_L(l) + |B| + 1$.
8. Output $\sigma$.

**Example**

Consider a tournament $T$ on four teams: $A_1, A_2, A_3$, and $A_4$. Assume that $A_1$ defeats $A_2$ but loses to $A_3$; $A_1$ and $A_4$ lose to $A_2$; and $A_2$ and $A_4$ lose to $A_3$. This information can be represented as a complete directed graph:

We now consider a simulation of one run of NRKotH as an illustration:

- We first pick a random team as the pivot, say it’s $A_2$.
- Note that $B = \{A_1\}$ is the set of teams that beat $A_2$ and $L = \{A_1, A_2\}$ is the set of teams that lose to $A_2$.
- We give $A_1$ rank $|B| + 1 = 2$.
- We run NRKotH on $B$. Since $|B| = 1$, we give $A_1$ rank 1.
- We run NRKotH on $L$. We first pick a random team as the pivot; say it’s $A_3$.
- Since $A_1$ defeats $A_2$, $A_1$ gets rank 1 and $A_2$ gets rank 2 in this sub-tournament.
- This translates to $A_1$ getting rank $1 + |B| + 1 = 3$ and $A_2$ getting rank $2 + |B| + 1 = 4$ in the original tournament.
- The final ranking is thus $(A_3, A_2, A_1, A_4)$.

If the prize vector was the Borda prize vector, the prizes awarded to $A_1, A_2, A_3$, and $A_4$ would be $1, 2/3, 0$, and $1/3$ respectively.

**Consistency Under Expectation**

For any team $u$ and tournament $T$, we define $\epsilon_T(u)$ to be the set of teams that $u$ defeats in tournament $T$ and $\sigma^k_r(u)$ to be the random variable that is the ranking of team $u$ under rule $r$, applied to $T$. A tournament rule $r$ is Consistent under Expectation if for all $n$, all tournaments $T$ on $n$ teams, and all $u$:

$$\epsilon_T(u) = n - |w_T(u)|$$

**Main Result I**

For any prize vector in $[0, 1]^n$, and any underlying tournament $T$, under the NRKotH tournament rule, no two teams can manipulate their match to gain expected prize money more than $1/3$. Mathematically, we can express this as

$$\epsilon^2_T(NRKotH) = \frac{1}{3} = \sigma^2_2$$

where $P$ denote the set of all prize vectors in $[0, 1]^n$. Moreover, this is the best possible guarantee for any Condorcet-Consistent tournament ranking rule.

**Main Result II**

NRKotH is Consistent under Expectation. As a consequence, for the Borda prize vector $\vec{p}$, no set of $k$ teams can manipulate any of their matches to gain any additional expected prize money. Mathematically, we can express this as

$$\alpha^k_r(NRKotH) = 0$$

for all $k \leq n$. Further, for the class of prize vectors $P$ $\epsilon$-close to the Borda prize vector (consisting of vectors $p$ where $p_j \in \left[\frac{1}{n-1} - \epsilon, \frac{1}{n-1} + \epsilon\right]$), we have that

$$\alpha^k_r(NRKotH) \leq 2k\epsilon$$

**Remarks**

- Though NRKotH is competitive even with the best Condorcet-Consistent tournament rule, and even with the best guarantee achievable just on $(1, 0, \ldots, 0)$, it achieves a significantly stronger fairness guarantee: it is Cover-Consistent.
- NRKotH is “equivalent” to the quicksort sorting algorithm. Tournament rules equivalent to the mergesort and bubblesort sorting algorithms are not consistent under expectation and do not satisfy Main Result II.

**References**
